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ANALYSIS OF DISC BRAKE NOISE USING A TWO-DEGREE-OF-FREEDOM MODEL

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Although a brake pad and disc have many modes of vibration, when it is unstable and hence noisy then frequently only a single mode of the complete system contributes to the vibration. In this condition, only a few modes are required to model the system. In this paper, a two-degree-of-freedom model is adopted where the disc and the pad are modelled as single modes connected by a sliding friction interface. Using this model, the interaction between the pad and the disc is investigated. Stability analysis is performed to show under what parametric conditions the system becomes unstable, assuming that the existence of a limit cycle represents the noisy state of the disc brake system. The results of this analysis show that the damping of the disc is as important as that of the pad. Non-linear analysis is also performed to demonstrate various limit cycles in the phase space. The results show that the addition of damping to either the disc or the pad alone may make the system more unstable, and hence noisy.

1. INTRODUCTION

Automotive brake noise has been investigated by many researchers over the last few decades, and an interesting review on the different types of brake noise has been presented by Crolla and Lang [1]. Most modelling work in this area has been based on either lumped parameter methods, (see for example references [2–7]), or finite element methods (see for example references [8–12]). Whatever the modelling method, the general approach adopted by many is the use of linear analysis to determine when the system is stable for various parametrical conditions. An unstable system suggests that the brake will be noisy and a stable system otherwise. As the system is linear then stability analysis is performed by

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finding the real part of eigenvalues of the characteristic equation of the system. If these are negative then the system is stable, but if they are positive then the system is unstable.

In a brake system the friction force is the main excitation mechanism. It is a distributed force between the brake-pad and brake-disc and varies over the contact area in a complex manner, being influenced, amongst other factors, by the dynamic behaviour of the pad and disc. Recently, a finite element model incorporating the distributed frictional force has been developed [10]. Ibrahim has also published a comprehensive review on "friction induced vibration" [13, 14].

In order to understand the fundamental mechanisms of brake noise, several different single-degree-of-freedom system models have been studied in terms of stability analysis, as described in references [2, 15]. These models have been used to illustrate the different causes of instability, for example a negative friction–velocity gradient or particular geometry. A negative friction–velocity gradient leads to 'stick–slip' vibration and is recognized as one of the prime causes of brake noise. This occurs because of the non-linearity of the frictional force–velocity characteristic, and is characterized by uniform motion (stick motion that accumulates energy in the system) followed by non-uniform motion (slip motion that dissipates the accumulated energy). Stick–slip vibration has been an area of active research for many years, and a notable publication in this area was on the study of dynamics of the bowed-string of a violin by McIntyre and Woodhouse [16]. Since then, many articles concerning stability analysis of stick–slip motion have been published (for example references [17, 18]). Other types of behaviour can be produced by friction, and Popp and Stelter [19] have shown that a simple single-degree-of-freedom system can exhibit rich dynamics—from limit cycle, to quasi-periodic and chaotic behaviour. A relatively new approach to modelling the friction force, which shows that instability is dependent on initial conditions, has been introduced by McMillan [20] who used a hysteretic friction model.

Once a brake system has become unstable the noise generated is often a single tone, with the system vibrating at a resonant frequency. Thus, in this paper a two-degree-of-freedom model of a disc-brake system is studied. The disc and pad are modelled by single modes, and are connected through a sliding friction interface. The aim of the paper is to use this simple model to investigate the effect of damping on the stick–slip vibration of a disc-brake system. The model is not intended to be a complete dynamic model of a brake (it does not, for example, allow a study of effects of the mode shapes of the component parts), but it can be usefully used to illustrate some of the characteristics of stick–slip vibration. Although stick–slip vibration alone is not the sole source of brake noise (judder, moan, groan, squeal, etc. [1]), a study of the effects of damping in stick–slip vibration may help in the understanding of disc-brake instabilities. An analysis of the simple model is carried out using the characteristic equation to find the conditions for instability with particular interest in the damping of the component parts. Following this, a non-linear analysis is conducted to investigate the more subtle aspects of the system's behaviour. A negative friction–velocity gradient is used to characterize the interface, and this leads to "stick–slip" vibration. The friction model used is the simplest possible model that still produces self-excited vibrations (including stick–slip motion) as the aim of the paper is to investigate the fundamental system dynamics rather than the response of the system to different friction interfaces. Although several two-degree-of-freedom models have been investigated in the literature, for example, where masses are serially connected (both masses on the massless moving belt) or where a single mass is considered with two co-ordinates (horizontal and vertical) on a massless rotating disc [13, 21–25], to the authors' knowledge the model proposed in this paper, which consists of two single-degree-of-freedom systems connected through a sliding friction interface, is new.

2. TWO-DEGREE-OF-FREEDOM MODEL AND LINEAR STABILITY ANALYSIS

Consider the model shown in Figure 1(a). It represents the pad and disc as single-degree-of-freedom systems that are connected together through a sliding friction interface. The system with subscript 1 denotes the pad, the system with subscript 2 denotes the disc, and m, k and c denote mass, stiffness and damping respectively. The motion of the first mass (m_1) may represent the tangential motion of the pad, and the second mass (m_2) may represent the in-plane motion of disc. The normal force acting on the interface is $N = PS$ where P is the pressure applied and S is the surface area of the interface. The resulting frictional for F_f is dependent upon the normal force and the dynamic coefficient of friction between the two sliding surfaces. The disc motion is the superposition of a constant imposed velocity v_0 and velocity $\dot{x}_d$, and the pad motion has velocity $\dot{x}_p$.

Stick–slip motion is usually described as a limit cycle in phase space, and requires non-linear analysis to determine the detailed behaviour of the system. However, since the existence (and hence noise) or non-existence of a limit cycle depends on the stability of equilibrium points, then linear analysis can be used to determine the stability of these points. To conduct this investigation a linear friction model for the interface is used, and this is shown as a function of the relative velocity v_r between the pad and the disc in Figure 1(b), where μ_s is the static coefficient of friction and $\mu(v_r)$ is the dynamic coefficient of friction. The function $\mu(v_r)$, which has a negative gradient, has been specifically chosen for its simplicity, although it is recognized that more complicated functions might give a more detailed description of the interface properties.

Provided that the relative velocity is always positive, the frictional force is related to the normal force by

$$F_f = N(\mu_s - \alpha v_r) = N(\mu_s - \alpha v_0) + N\alpha(\dot{x}_p - \dot{x}_d) \tag{1}$$

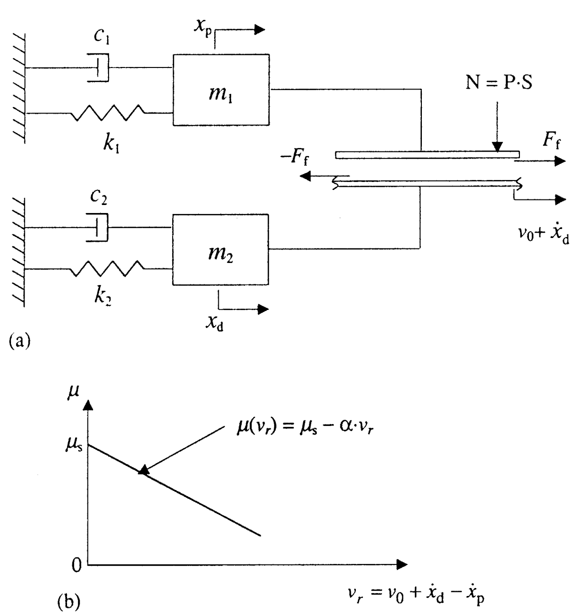

Figure 1. Two-degree-of-freedom model of a disc brake system. (a) Two-degree-of-freedom model; (b) dynamic friction coefficient.

and the equations of motion can thus be written as

$$m_1 \ddot{x}_p + c_1 \dot{x}_p - N\alpha(\dot{x}_p - \dot{x}_d) + k_1 x_p = N(\mu_s - \alpha v_0),$$

$$m_2 \ddot{x}_d + c_2 \dot{x}_d - N\alpha(\dot{x}_d - \dot{x}_p) + k_2 x_d = -N(\mu_s - \alpha v_0). \tag{2}$$

Note that the frictional forcing term is split into two parts; one is associated with a state variable, namely damping, and the other can be considered as an independent external force directly related to the dynamic friction coefficient. It can be seen in equation (2) that the term $N\alpha$ acts as negative damping, and is the only term connecting the pad and disc. For stability analysis, where the forcing term on the right side of equation (2) is not considered, the characteristic equation becomes

$$det\begin{bmatrix} \lambda^2 + c_{11}\lambda + k_{11} & c_{12}\lambda \\ c_{21}\lambda & \lambda^2 + c_{22}\lambda + k_{22} \end{bmatrix} = 0, \tag{3}$$

where $c_{11} = (c_1 - N\alpha)/m_1$, $c_{22} = (c_2 - N\alpha)/m_2$, $c_{12} = N\alpha/m_1$, $c_{21} = N\alpha/m_2$, $k_{11} = k_1/m_1$, $k_{22} = k_2/m_2$.

Equation (3) is a fourth order polynomial of the form $\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0$, and therefore from the Routh criterion [26], the conditions for instability are

$$a_1 < 0, \text{ or } a_2 < 0 \text{ or } a_3 < 0, \text{ or } a_4 < 0 \text{ or } a_1 a_2 - a_3 < 0 \text{ or } a_1 a_2 a_3 - a_1^2 a_4 - a_3^2 < 0, \tag{4}$$

where

$$a_1 = c_{22} + c_{22} = \frac{m_2(c_1 - N\alpha) + m_1(c_2 - N\alpha)}{m_1 m_2},$$

$$a_2 = c_{11}c_{12} - c_{12}c_{21} + k_{11} + k_{22} = \frac{(c_1 - N\alpha)(c_2 - N\alpha) - (N\alpha)^2 + m_1 k_2 + m_2 k_1}{m_1 m_2},$$

$$a_3 = k_{11}c_{22} + k_{22}c_{11} = \frac{k_1(c_2 - N\alpha) + k_2(c_1 - N\alpha)}{m_1 m_2},$$

$$a_4 = k_{11}k_{22} = \frac{k_1 k_2}{m_1 m_2}.$$

To examine the stability of the system for various conditions some simulations are presented. Since $N\alpha$ is the most important parameter and has the same units as the damping coefficient, it is normalized with respect to the damping of the pad (c_1). Figure 2 shows the regions of stability/instability for various values of $N\alpha$ and other normalized system parameters.

It can be seen from examining Figures 2(a) and 2(b) that when $m_1 \approx m_2$ and when $k_1 \approx k_2$ smaller values of $N\alpha/c_1$ will make the system unstable and hence noisy. This implies that when the natural frequencies of the pad and the disc are the same, then for a given set of operating conditions the system will be less stable. Figures 2(a2) and 2(b2) suggest that increasing damping in either the disc or the pad will have a beneficial effect. It should also be noted from Figures 2(a) and 2(b) that the $N\alpha$ term should never be greater than c_1 or c_2 for a stable condition. The general conclusion from this analysis is that for maximum stability the natural frequencies of the disc and the pad should be well separated and be well damped.

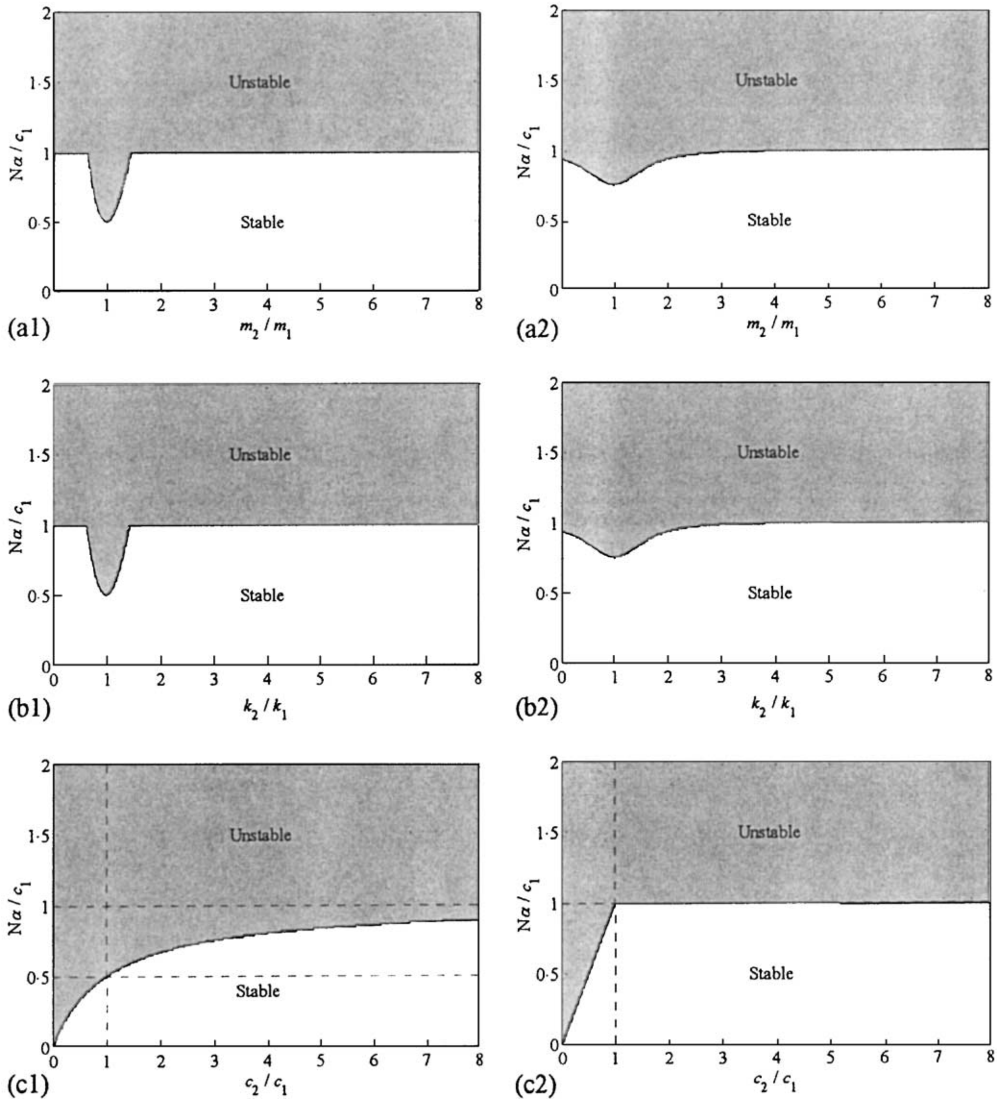

Figure 2. Stability area of the "$N\alpha$" term versus various combinations of system parameters. (a1) "$N\alpha$" term versus mass ($k_2 = k_1$, $c_2 = c_1$), (a2) "$N\alpha$" term versus mass ($k_2 = k_1$, $c_2 = 3c_1$), (b1) "$N\alpha$" term versus stiffness ($m_2 = m_1$, $c_2 = c_1$), (b2) "$N\alpha$" term versus stiffness ($m_2 = m_1$, $c_2 = 3c_1$). (c1) "$N\alpha$" term versus "c_2" ($m_2 = m_1$, $k_2 = k_1$), (c2) "$N\alpha$" term versus "c_2" ($m_2 = m_1$, $k_2 = 5k_1$).

A stability criterion for the case when the ratio of two natural frequencies is large, say more than 2, is

$$\min(c_1, c_2) > N\alpha \quad \text{when the natural frequencies are significantly different,} \tag{5}$$

where $\min(c_1, c_2)$ denotes the minimum value. Figures 2(c1) and 2(c2) show the plots of $N\alpha/c_1$ as a function of c_2/c_1 for the situations when the natural frequencies are coincident and when they are different. These figures further verify the above result, i.e., the brake is more likely to be noisy when natural frequencies of the pad and disc are similar. They also show that the system generally becomes more stable as the damping increases. From Figures 2(c1) and 2(c2) it can be seen that c_2/c_1 is never smaller than twice $N\alpha/c_1$ while maintaining its stability. Thus, it is guaranteed that the system is *always* stable provided that

$$\min(c_1, c_2) > 2N\alpha. \tag{6}$$

The criteria given in equations (5) and (6) imply that no matter how much damping is added to the system on one side of the sliding interface, the system can become unstable unless an appropriate level of damping is added to the other side.

In many disc-brake installations, damping is only added to the pad because it is relatively easy to add damping to the pad rather than the disc. Sometimes this is effective, but anecdotal evidence suggests that in some cases brake noise becomes worse after damping treatment. The above results offer an explanation why this might be so, and this is further investigated in the next section.

3. NON-LINEAR ANALYSIS

Whilst the stability analysis conducted in the previous section is useful, it does not provide detailed information on the non-linear dynamical behaviour of the system. For example, it is possible for linear analysis to predict an unstable system, but the resulting limit cycle may be very small and hence the noise generated would be inaudible. Non-linear analysis can provide information on the size of a limit cycle and hence whether a particular instability is a problem. The friction–velocity relationship used in this section is similar to that used in section 2, but it allows a negative relative velocity. It is shown in Figure 3, where it can be seen that there is a discontinuity at zero relative velocity. This causes highly non-linear behaviour and produces stick–slip motion. When analyzing this non-linear system, difficulties arise from the discontinuity in the friction force. This can be overcome by either using the "smoothing method" or the "switching method" [27], and in this paper, the switching method is used.

The motion of the system is governed by the static friction force in the stick motion and by a velocity-dependent friction force in the slip motion. For the stick mode, the static friction force is limited by the maximum state friction force, i.e., $|F_s| \leqslant \mu_s N$, and is balanced with the reaction forces acting on the masses. Considering the relative motion between the two masses the static friction force can be written as

$$F_s = k_1 x_p + c_1 \dot{x}_p - k_2 x_d - c_2 \dot{x}_d \tag{7}$$

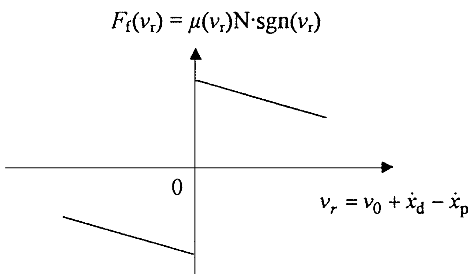

Figure 3. Friction force, $F_f(v_r)$.

and the frictional force can be described by

$$F_f = \begin{cases} \min(|F_s|, \mu_s N)\,\mathrm{sgn}(F_s) & \text{for } v_r = 0 \text{ stick}, \\ \mu(v_r) N\,\mathrm{sgn}(v_r) & \text{for } v_r \neq 0 \text{ slip}. \end{cases} \tag{8}$$

For numerical analysis, the friction force is switched appropriately according to the type of motion, and a small region ε of the relative velocity is defined, i.e., $|v_r| < \varepsilon$, where $\varepsilon \ll v_0$. The equations of motion for the system are given by

$$m_1 \ddot{x}_p + c_1 \dot{x}_p + k_1 x_p = F_f(v_r) - F_f(v_0),$$

$$m_2 \ddot{x}_d + c_2 \dot{x}_d + k_2 x_d = -\,[F_f(v_r) - F_f(v_0)], \tag{9}$$

where, $v_r = v_0 + \dot{x}_d - \dot{x}_p$, and $F_f(v_0) = N(\mu_s - \alpha v_0)$ is introduced to compensate for the offset. Equation (9) can be rewritten as four first order differential equations to facilitate computation as

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\frac{c_1}{m_1} x_2 - \frac{k_1}{m_1} x_1 + \frac{1}{m_1}(F_f(v_r) - F_f(v_0)),$$

$$\dot{x}_3 = x_4, \qquad \dot{x}_4 = -\frac{c_2}{m_2} x_4 - \frac{k_1}{m_2} x_3 - \frac{1}{m_2}(F_f(v_r) - F_f(v_0)), \tag{10}$$

where $x_p = x_1$, $\dot{x}_p = \dot{x}_1 = x_2$, $x_d = x_3$, $\dot{x}_d = \dot{x}_3 = x_4$. Equation (10) can be solved numerically to determine the attractors of the system for various system parameters. Because the two masses (pad and disc) are acted on by the same friction force, the resulting motion of both pad and disc are the same if $m_1 = m_2$, $c_1 = c_2$ and $k_1 = k_2$. In this case, the dynamics are similar to the single-degree-of-freedom system described in reference [19], i.e., stick–slip limit cycle motion is dominant provided that damping is sufficiently small. This case is considered first to demonstrate the influence of parameters related to the friction force, i.e., α, N and v_0. For this simulation the system parameters are arbitrarily set to $m_1 = m_2 = k_1 = k_2 = 1$, $c_1 = c_2 = 0{\cdot}01$, and the static friction coefficient is set to $\mu_s = 0{\cdot}6$. This stick–slip motions for various values of friction parameters are shown in Figure 4. In this figure, only pad motion is shown since the disc motion is almost identical.

From Figure 4(a), it can be seen that a steady limit cycle occurs if α is very small. This is probably because the dynamic friction coefficient is similar to the static friction coefficient in this case. As α is increased, however, a stick–slip limit cycle occurs and the size of limit cycle increases. This shows that brake noise will probably become worse when the negative gradient of the dynamic friction coefficient increases. Similar results are obtained for the case of normal force "N" and the velocity "v_0" as shown in Figures 4(b) and 4(c), respectively, i.e., the size of limit cycle increases as increasing the values of N and v_0.

The above results further verify the importance of the term $N\alpha$ described in the previous section. Also, it is shown that the input velocity v_0 affects the slope of dynamic friction coefficient, thus affecting the size of the limit cycle. The parameters α, N and v_0 may be difficult to change when designing a brake system. It may be easier to change the system parameters (mass, damping and stiffness), and thus the effects of these parameters on the dynamic behaviour of the system is investigated, with particular emphasis on the damping parameter.

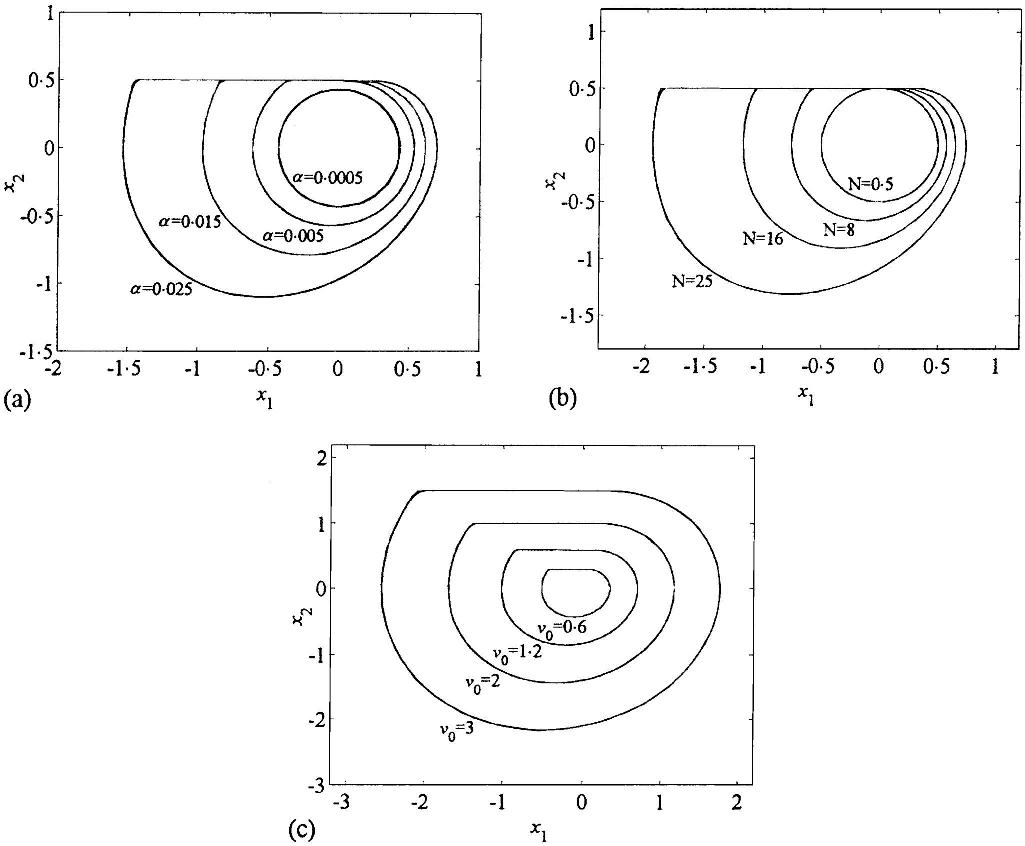

Figure 4. Limit cycle motions for various values of friction parameters. (a) Motions of the pad for various values of "α" (where $N = 10$ and $v_0 = 1$). (b) Motions of the pad for various values of "N" (where $\alpha = 0{\cdot}012$, and $v_0 = 1$). (c) Motions of the pad for various values of "v_0" (where $N = 10$ and $\alpha = 0{\cdot}012$).

Various combinations of system parameters are considered while the friction parameters are set to $N = 10$, $\alpha = 0{\cdot}012$ and $v_0 = 1$ for all the cases presented. The first criterion in section 2, "$\min(c_1, c_2) > N\alpha$" is verified for a large difference in two natural frequencies, by observing the motions for two cases obtained with system parameters, $m_1 = m_2 = 1, k_1 = 1$, $k_2 = 2$ and $c_1 = c_2 = 0{\cdot}11$ and $m_1 = m_2 = 1, k_1 = 1, k_2 = 2$ and $c_1 = c_2 = 0{\cdot}13$ respectively. Note that the value of $N\alpha$ is 0·12, in this case. Also, note that both damping parameters must be larger than $N\alpha$ to meet the criterion. Since the qualitative motions of the pad and the disc are similar, only the motions of the pad are presented in the results shown in Figure 5.

As shown in Figure 5(a), when the damping parameters are smaller than $N\alpha$, the motion is a steady limit cycle showing that there is an unstable equilibrium point. If the values are greater than $N\alpha$, the motion gradually dies away to a stable equilibrium point, and the time history for this case is shown in Figure 5(b). Consider now the case when the natural frequencies of the pad and disc are the same, and the parameters are $m_1 = m_2 = k_1 = 1$ and $c_1 = c_2 = 0{\cdot}13$, i.e., the same damping parameters as with the case of Figure 5(b). The results are shown in Figure 5(c) where it can be seen that the motion now becomes a stick–slip limit cycle. Whilst conducting the simulations, it was found that the motion of the system

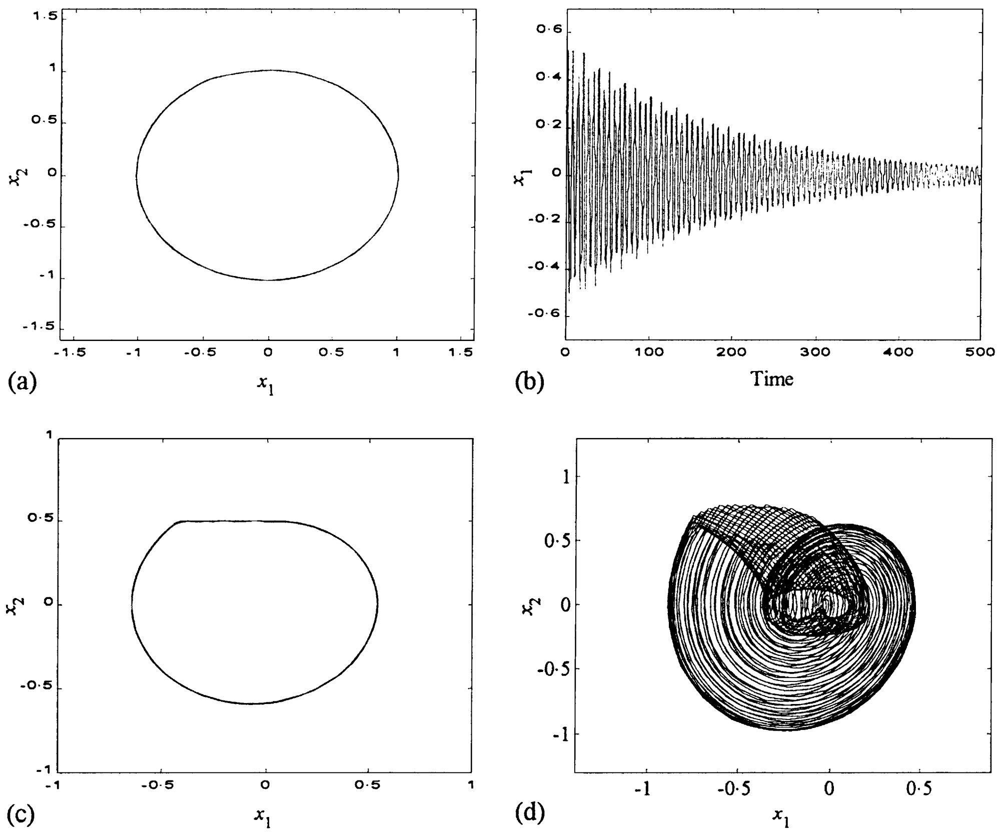

Figure 5. Various pad motions for $N = 10$, $\alpha = 0{\cdot}012$ and $v_0 = 1$. (a) Limit cycle motion of the pad ($m_1 = m_2 = 1$, $k_1 = 1$, $k_2 = 2$, $c_1 = c_2 = 0{\cdot}11$). (b) Time history of the pad ($m_1 = m_2 = 1$, $k_1 = 1$, $k_2 = 2$, $c_1 = c_2 = 0{\cdot}13$). (c) Stick–slip motion of the pad ($m_1 = m_2 = 1, k_1 = k_2 = 1, c_1 = c_2 = 0{\cdot}13$). (d) Chaotic motions of the pad ($m_1 = 0{\cdot}5$, $m_2 = k_1 = k_2 = 1$, $c_1 = c_2 = 0{\cdot}01$).

appeared to become chaotic in certain conditions, an example of which is shown in Figure 5(d). Note that this figure is after the transient motion dies out. Further analysis would be needed to confirm that this is actually chaotic. However, the analysis of such behaviour is not the aim of the paper, and so is not considered further.

Because damping plays an important role, its effects on the limit cycles are further examined. For this study, mass and stiffness parameters are fixed so that the pad and the disc have the same natural frequencies, i.e., $m_1 = m_2 = k_1 = k_2 = 1$, and the friction parameters are as before. The damping parameters (c_1 and c_2) are then varied to examine the limit cycle behaviour of both pad and disc, the results of which can be seen in Figures 6 and 7.

It is found that the size of the limit cycles decrease as the damping of both pad and disc increase simultaneously, and finally end up as fixed points when the damping parameters are sufficiently large as shown in Figure 6. It should be noted that in these simulations the value of $2N\alpha$ is 0·24 and $c_1 = c_2 = 0{\cdot}22$ for the limit cycles in Figures 6(a1) and 6(b1) and $c_1 = c_2 = 0{\cdot}26$ for the phase plane plots in Figures 6(b1) and 6(b2). Thus the second criterion in section 2, i.e., "$\min(c_1, c_2) > 2N\alpha$" for two natural frequencies the same, is validated.

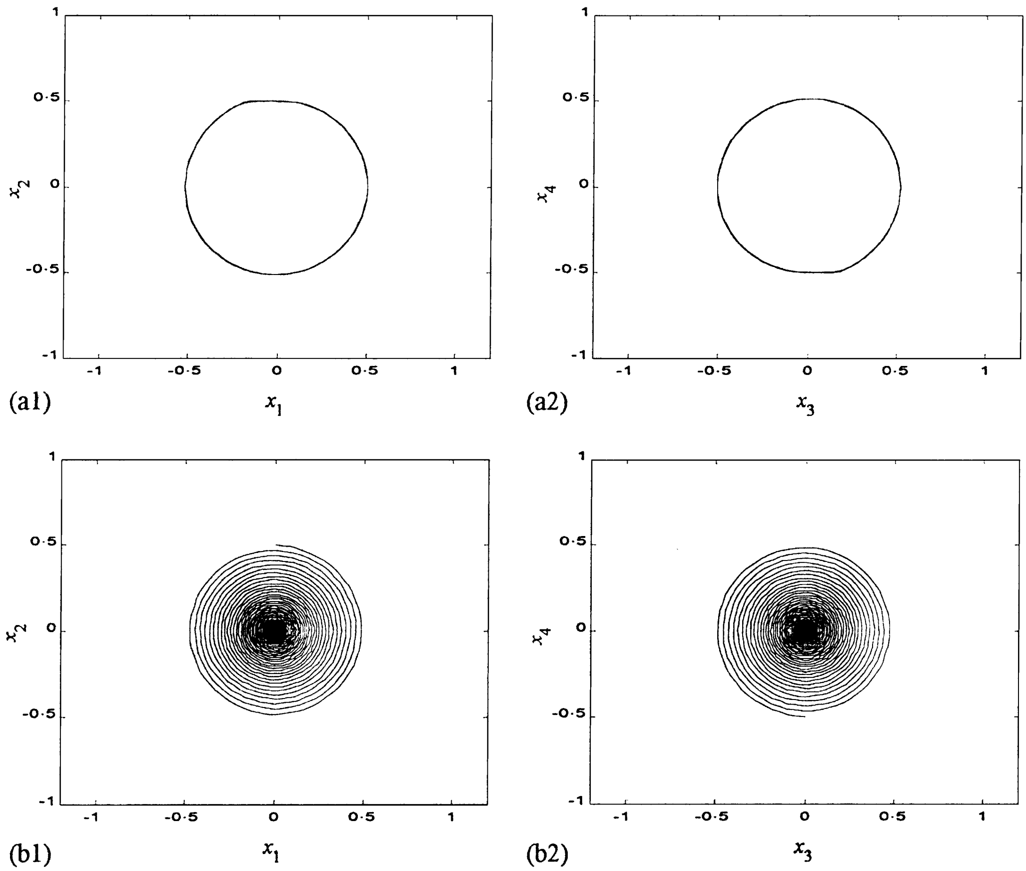

Figure 6. Motions when both damping are increased, where "$m_1 = m_2 = k_1 = k_2 = 1$". (a1) Stick–slip motion of the pad ($c_1 = c_2 = 0{\cdot}22$). (a2) Stick–slip motion of the disc ($c_1 = c_2 = 0{\cdot}22$). (b1) Stable motion of the pad ($c_1 = c_2 = 0{\cdot}26$). (b2) Stable motion of the disc ($c_1 = c_2 = 0{\cdot}26$).

However, if the damping is increased in the system only on one side of the interface, for example in the pad, the size of limit cycle corresponding to the pad decreases, whereas the limit cycle of the disc increases. This is demonstrated in Figure 7, where the damping of the disc is fixed, $c_2 = 0{\cdot}01$, and the damping of pad is gradually increased. It is also found that the system does not go to a fixed point no matter how much damping is added.

4. CONCLUSIONS

A two-degree-of-freedom model has been used to investigate the basic mechanisms of an instability that is one of the causes of disc brake noise. The model has also been used to demonstrate the conditions necessary for preventing the instability. The analysis suggests that when the natural frequencies of a pad and disc are in close proximity then a brake is more likely to be noisy. It has also been found that the amount and distribution of damping in the system is a key factor. The damping of the pad and the disc are of equal importance in the prevention of an instability and hence noise. Non-linear analysis has been conducted to investigate the detailed dynamical behaviour of the model for various combinations of both

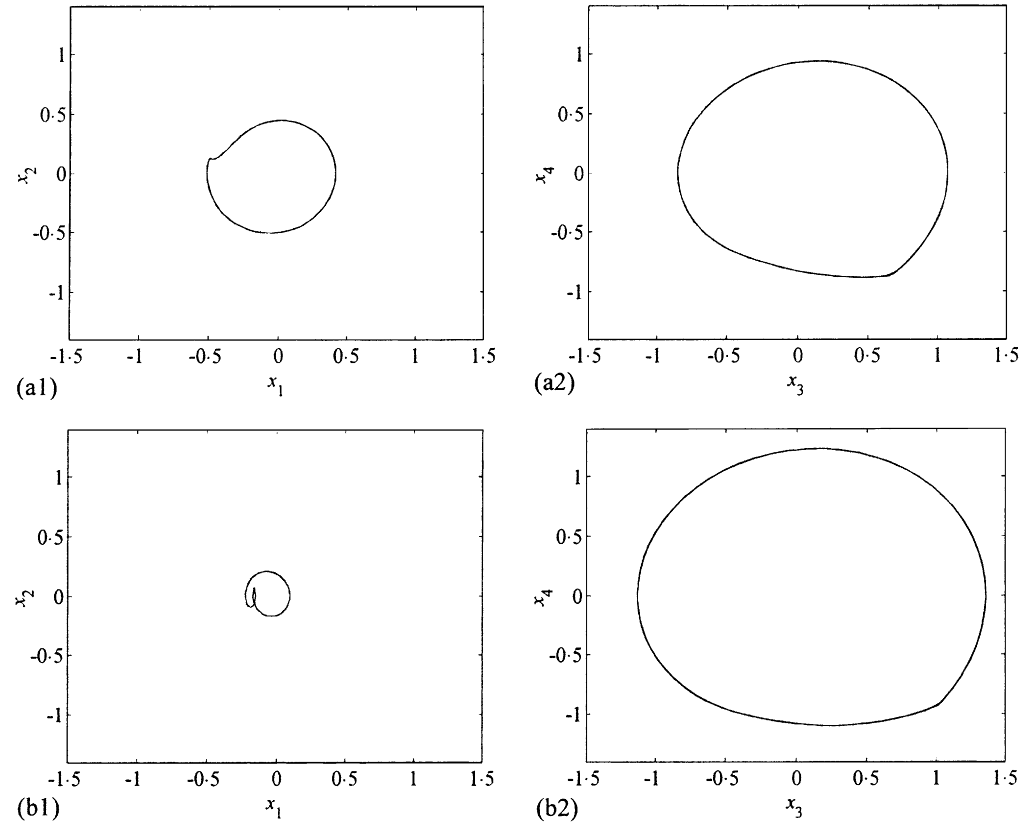

Figure 7. Motions when only one damping is increased, where "$m_1 = m_2 = k_1 = k_2 = 1$". (a1) Limit cycle motion of the pad ($c_1 = 0{\cdot}16$, $c_2 = 0{\cdot}01$). (a2) Limit cycle motion of the disc ($c_1 = 0{\cdot}16$, $c_2 = 0{\cdot}01$). (b1) Limit cycle motion of the pad ($c_1 = 0{\cdot}56$, $c_2 = 0{\cdot}01$). (b2) Limit cycle motion of the disc ($c_1 = 0{\cdot}56$, $c_2 = 0{\cdot}01$).

friction parameters and system parameters. The results further confirm the importance of damping on the squeal noise, and have shown the *increasing* damping of either the disc or the pad alone can potentially have detrimental effects on the system stability.

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